University of Puerto Rico
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Qualifying Examination

Area: Nonlinear Programming   Date: 1 May 2003

Solve any three of the following five problems.

1. Let f be a convex function on a convex set Ω. Then prove the following:

   (a) (50%) The set $\Gamma_c = \{\mathbf{x} : \mathbf{x} \in \Omega, f(\mathbf{x}) \leq c\}$ is convex for every real number c.

   (b) (50%) The set Γ where f achieves its minimum is convex, and any relative minimum of f is a global minimum.

2. Consider the method of steepest descent applied to the quadratic case

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \frac{\mathbf{g}_k^T \mathbf{g}_k}{\mathbf{g}_k^T Q \mathbf{g}_k} \mathbf{g}_k, \tag{1}$$

   where $\mathbf{g}_k = Q\mathbf{x}_k - \mathbf{b}$, and Q is a symmetric positive definite $n \times n$ matrix.

   (a) (35%) Show that the iterative process (**??**) satisfies

$$E(\mathbf{x}_{k+1}) = \left\{ 1 - \frac{(\mathbf{g}_k^T \mathbf{g}_k)^2}{(\mathbf{g}_k^T Q \mathbf{g}_k)(\mathbf{g}_k^T Q^{-1} \mathbf{g}_k)} \right\} E(\mathbf{x}_k),$$

   where

$$E(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T Q (\mathbf{x} - \mathbf{x}^*), \tag{2}$$

   and $\mathbf{x}^*$ is the unique minimum point of

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x} - \mathbf{b}^T \mathbf{x}. \tag{3}$$

(b) (35%) Without assuming the Kantorovich inequality, show that for any vector $\mathbf{x}$ there holds

$$\frac{(\mathbf{x}^T\mathbf{x})^2}{(\mathbf{x}^TQ\mathbf{x})(\mathbf{x}^TQ^{-1}\mathbf{x})} \geq \frac{1}{\mathrm{Cond}_*(Q)},$$

where $\mathrm{Cond}_*(Q) = \dfrac{\lambda_n}{\lambda_1}$, and λ_1 and λ_n are the smallest and largest eigenvalues of Q.

(c) (30%) Show that for any $\mathbf{x}_0 \in \mathbb{R}^n$ the method (**??**) converges to the unique minimum point $\mathbf{x}^*$ of (**??**). Furthermore, prove that there holds at every step k

$$E(\mathbf{x}_{k+1}) \leq \left(\frac{\mathrm{Cond}_*(Q) - 1}{\mathrm{Cond}_*(Q)}\right) E(\mathbf{x}_k).$$

3. (a) (35%) Show that the point $\mathbf{x}_{k+1}$ generated by the conjugate gradient method for the quadratic problem (**??**) satisfies

$$E(\mathbf{x}_{k+1}) = \min_{P_k} \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}^*)^T Q \left[I + QP_k(Q)\right]^2 (\mathbf{x}_0 - \mathbf{x}^*),$$

where E is defined in (**??**) and the minimum is taken with respect to all polynomials P_k of degree k.

(b) (35%) Prove that

$$E(\mathbf{x}_{k+1}) \leq \max_{\lambda_i} \left[1 + \lambda_i P_k(\lambda_i)\right]^2 E(\mathbf{x}_0),$$

for any polynomial P_k of degree k, where the maximum is taken over all eigenvalues λ_i of Q.

(c) (30%) Use the result in part **??** to show that the conjugate gradient method will find the solution within n iterations.

4. Let H be an invertible $n \times n$ matrix. Consider the iterative method to approximate the inverse matrix of H, defined by

$$H_{k+1} = H_k + \beta_k \mathbf{z}_k \mathbf{z}_k^T,$$

such that

$$H_{k+1}\mathbf{q}_k = \mathbf{p}_k,$$

where H_0 (an $n \times n$ symmetric matrix), $\mathbf{q}_0, \mathbf{p}_0 \in \mathbb{R}^n$ are given.

(a) (50%) Find values for $\beta_k \in \mathbb{R}$ and $\mathbf{z}_k \in \mathbb{R}^n$.

(b) (50%) Assuming that $\mathbf{q}_k^T(\mathbf{p}_k - H_k\mathbf{q}_k) > 0$ for all k and H_0 is positive definite, show that H_k is positive definite for all k.

5. Consider the problem

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{subject to} & \mathbf{h}(\mathbf{x}) = 0, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$. Let $\mathbf{x}^*$ be a regular point of the constraints $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ and a minimum point of f subject to these constraints.

(a) (50%) Show that all $\mathbf{y} \in \mathbb{R}^n$ satisfying

$$\nabla \mathbf{h}(\mathbf{x}^*)\mathbf{y} = \mathbf{0}$$

must also satisfy

$$\nabla f(\mathbf{x}^*)\mathbf{y} = 0.$$

(b) (50%) Show that there is a $\lambda \in \mathbb{R}^m$ such that

$$\nabla f(\mathbf{x}^*) + \lambda^T \nabla \mathbf{h}(\mathbf{x}^*) = \mathbf{0}.$$