University of Puerto Rico
Collage of Natural Sciences
Department of Mathematics
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Qualifying Examination

Area: Optimization **Date:** Wednesday, 10 September 2014

Solve any three of the following five problems.

1. Consider the problem
$$\min f(x,y)$$
where $f(x,y) = 2x^2 + 2y^2 - 2xy - 10x + 5y + 30$.

   (a) (30%) Find a critical point (x^*, y^*) of $f(x,y)$ and show that this point is a global minimum.

   (b) (40%) What would be the rate of convergence of steepest descent for this problem?

   (c) (30%) Starting at $x = y = 0$, how many steepest descent iterations would it take to reduce the function value
$$E(x,y) = \frac{1}{2}(x - x^*, y - y^*)Q(x - x^*, y - y^*)^T$$
to 10^{-12}? Here Q is the Hessian matrix of f. (Please answer this question without actually doing the iterations.)

2. Let H be a nonsingular $n \times n$ matrix.

   (a) (50%) Consider the iterative method to approximate the inverse matrix of H, defined by
$$H_{k+1} = H_k + \beta_k \mathbf{z}_k \mathbf{z}_k^T$$
such that
$$H_{k+1}\mathbf{q}_k = \mathbf{p}_k$$
where H_0, $\mathbf{q}_0$ and $\mathbf{p}_0$ are given. Find the values of $\beta_k \in \mathbb{R}$ and $\mathbf{z}_k \in \mathbb{R}^n$ that correspond to the rank one update formula.

   (b) (50%) Show that starting with the rank one update formula for the inverse Hessian obtained in part (a), forming the complementary formula, and then taking the inverse restores the original formula.

3. Consider the problem

$$\min\ (x^3 - y)^2 + 2(y - x)^4.$$

(a) (20%) Show that the problem has multiple optima.

(b) (40%) Write a method to solve this problem by using Newton's method with a line search. Explain the limitations, if any, of this approach.

(c) (40%) Write a method to solve this problem by using the simulated annealing method. Is there any limitation with this approach?

4. Consider the problem

$$\begin{array}{ll} \min & \frac{1}{2}\|A\mathbf{x} - \mathbf{b}\|_2^2 \\ \text{subject to} & \|D\mathbf{x}\|_2 \leq \epsilon, \end{array}$$

where A is an $m \times n$ real matrix, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, D is an $n \times n$ real nonsingular matrix, $n \leq m$, and $\epsilon > 0$.

(a) (20%) Rewrite the constrained problem as an unconstrained problem by using a suitable penalty function.

(b) (40%) Write the solution of the resultant unconstrained problem as a function of the penalty coefficient and show that the unconstrained problem has a unique solution for a fixed penalty coefficient.

(c) (40%) By using the expression obtained in part (b), choose a suitable penalty coefficient to solve the constrained problem with the following data

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad D = I, \quad \text{and} \quad \epsilon = 0.1.$$

5. Consider the following constrained problem:

$$\min\ (2y_1 + 2y_2)$$

$$\text{subject to} \quad \left\{ \begin{array}{rcl} x_1^2 + y_1^2 & = & 25, \\ (x_1 - x_2)^2 + (y_1 - y_2)^2 & = & 4, \\ (4 - x_2)^2 + y_2^2 & = & 1. \end{array} \right.$$

(Note that there are four variables in this problem!)

(a) (40%) Write down or derive the first order necessary conditions for a solution to this problem. (To fix the signs of the multipliers, write the Lagrangian as $f - \sum_i \lambda_i g_i$ where the λ_i's are the multipliers.)

(b) (20%) Verify that $x_1 = x_2 = 4$, $y_1 = -3$, $y_2 = -1$ satisfies the first order necessary conditions for a minimum with multipliers

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{2}, \quad \lambda_3 = -2.$$

(c) (40%) Prove that the point of part (b) is actually a (relative) minimum for our problem.