University of Puerto Rico
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Qualifying Examination

Area: Nonlinear Programming   Date: 30 November 2004

Solve any three of the following five problems.

1. (a) (50%) Let γ be a monotone nondecreasing function of a single variable (that is, $\gamma(r) \leq \gamma(r')$ for $r' > r$) which is also convex; and let f be a convex function defined on a convex set Ω. Show that the function $\gamma(f)$ defined by $\gamma(f)(\mathbf{x}) = \gamma[f(\mathbf{x})]$ is convex on Ω.

   (b) (50%) Let f be twice continuously differentiable on a region $\Omega \subset \mathbb{R}^n$. Show that a sufficient condition for a point $\mathbf{x}^*$ in the interior of Ω to be a relative minimum point of f is that $\nabla f(\mathbf{x}^*) = \mathbf{0}$ and that f be locally convex at $\mathbf{x}^*$.

2. Consider the problem

$$\min f(x, y)$$

   where $f(x, y) = 2x^2 + 2y^2 - xy - 15x + 15y + 15$.

   (a) (40%) Find a point (x^*, y^*) satisfying the first-oreder necessary conditions for a solution and show that this point is a global minimum.

   (b) (30%) What would be the rate of convergence of steepest descent for this problem?

   (c) (30%) Starting at $x = y = 0$, how many steepest descent iterations would it take (at most) to reduce the function value

$$E(x, y) = \frac{1}{2}(x - x^*, y - y^*)Q(x - x^*, y - y^*)^T$$

   to 10^{-11}? Q is the Hessian matrix of f.

3. Consider the function f defined as

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{b}^T\mathbf{x}, \tag{1}$$

where Q is a real $n \times n$ symmetric and positive definite matrix and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. Let $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$ be conjugate vectors with respect to Q, i.e., $\mathbf{p}_i^T Q\mathbf{p}_j = 0$, for $i \neq j$. Let P be the matrix with columns $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$.

(a) (30%) Show that $P^T QP = D$ where D is a diagonal matrix.

(b) (30%) Show that with the change of variables $\mathbf{x} = P\mathbf{y}$, the function f defined in (**??**) becomes

$$\frac{1}{2}\mathbf{y}^T D\mathbf{y} - (P^T\mathbf{b})^T\mathbf{y}. \tag{2}$$

(c) (40%) Show now that the conjugate direction method with directions $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$ applied to (**??**) is equivalent to the conjugate direction method with directions $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ applied to (**??**). (Here $\mathbf{e}_i$ is the unit vector with 1 in the i^{th} position and the other components equal to zero.)

4. Consider the iterative process for minimizing the function f defined in (**??**)

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha_k S_k \mathbf{g}_k, \tag{3}$$

where S_k is a real symmetric $n \times n$ matrix,

$$\mathbf{g}_k = Q\mathbf{x}_k - \mathbf{b},$$

and

$$\alpha_k = \frac{\mathbf{g}_k^T S_k \mathbf{g}_k}{\mathbf{g}_k^T S_k Q S_k \mathbf{g}_k}.$$

Note that if $S_k = Q^{-1}$ then we obtain Newton's method, while if $S_k = I$ then we get steepest descent. Moreover, define

$$E(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T Q(\mathbf{x} - \mathbf{x}^*),$$

where $\mathbf{x}^*$ is the unique minimum point of f.

(a) (60%) Show that for the algorithm (**??**) there holds at every step k

$$E(\mathbf{x}_{k+1}) \leq \left(\frac{B_k - b_k}{B_k + b_k}\right)^2 E(\mathbf{x}_k),$$

where b_k and B_k are the smallest and largest eigenvalues of the matrix $S_k Q$, respectively.

(b) (40%) Now, in algorithm (**??**), let $S_k = I - V$, where I is the identity and V is a matrix whose eigenvalues satisfy $|\lambda_i| \leq e < 1, 1 \leq i \leq n$. With the provided information, what is the best bound on the rate of convergence of this algorithm?

5. Consider the quadratic program

$$\begin{aligned}
&\min \frac{1}{2}\mathbf{x}^T Q \mathbf{x} - \mathbf{b}^T \mathbf{x} \\
&\text{subject to } A\mathbf{x} = \mathbf{c}.
\end{aligned}$$

Prove that $\mathbf{x}^*$ is a local minimum point if and only if it is a global minimum point. (No convexity is assumed.)