University of Puerto Rico
College of Natural Sciences
Department of Mathematics
Río Piedras Campus

MS Qualifying Examination

Area: Computational Analysis **Date: 2 April 2018**

Solve any three of the following five problems.

1. Let A be a real symmetric positive definite matrix and consider solving the linear system
$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

   (a) (5 points) Prove that solving the linear system (1) is equivalent to computing
$$\mathbf{x} = \sum_{i=1}^{n} \left(\frac{\alpha_i}{\lambda_i}\right) \mathbf{u}_i,$$
   where λ_i is eigenvalue of A, $\mathbf{u}_i$ is the corresponding eigenvector, and the α_i's are the coordinates of $\mathbf{b}$ for the basis of the eigenvectors of A.

   (b) (5 points) Consider the following example
$$\begin{bmatrix} 1001 & 1000 \\ 1000 & 1001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
   use part (a) to verify that, when $\mathbf{b} = [2001, 2001]^{\mathrm{T}}$, the small change $\delta\mathbf{b} = [1, 0]^{\mathrm{T}}$ produces large variations in the solution. Explain why that happens.

2. QR decomposition.

   (a) (4 points) For a given vector $\mathbf{x} \in \mathbb{R}^n$ and r a positive integer number, find the Householder matrix H, which is an orthogonal matrix such that $H\mathbf{x}$ annihilates the elements below x_r and keeps unchanged the elements above x_r. Further, find the expression for x_r.

   Now, consider the following matrix and column vector
$$A = \begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}.$$

   (b) (3 points) Use Householder matrices to produce the QR factorization of A, where Q is an orthogonal matrix and R is an upper trapezoidal matrix.

   (c) (3 points) Using the QR factorization from part (a), solve the linear system $A\mathbf{x} = \mathbf{b}$ in the least-squares sense and report the square of the residual norm.

3. Gauss-Jacobi iterative method.

   (a) (3 points) An $n \times n$ real matrix A is strictly diagonal dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}} |a_{ij}| \geq 0.$$

   Prove that for a strictly diagonal dominant matrix the Gauss-Jacobi method converges to the solution $\mathbf{x}^*$ of (1) for any initial $\mathbf{x}^{(0)}$.

   (b) (4 points) Consider the matrix

$$H = \begin{bmatrix} 0 & -1/3 & -1/3 & 0 \\ -1/3 & 0 & -1/3 & -1/3 \\ -1/3 & -1/3 & 0 & -1/3 \\ 0 & -1/3 & -1/3 & 0 \end{bmatrix}.$$

   By using the Gerschgorin Theorem, the transformation

$$B = D^{-1} H D,$$

   for a suitable diagonal matrix D, and noting the relationship between the matrices H and B, prove that $\rho(H) < 1$.

   (c) (2 points) Prove that the Gauss-Jacobi method converges for the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}.$$

   (d) (1 point) The matrix A in 3(c) is a symmetric diagonally dominant (SDD) matrix. Does Gauss-Jacobi converge for all SDD matrices? Prove your answer.

4. Nonlinear equations.

   (a) (5 points) For the approximation of the zeros of the function

$$f(x) = \frac{2x^2 - 3x - 2}{x - 1},$$

   consider the following fixed-point methods:

$$x^{(k+1)} = g(x^{(k)}), \quad \text{where} \quad g(x) = x^2 + \frac{x}{2} + \frac{1}{2}; \tag{2}$$

   and

$$x^{(k+1)} = h(x^{(k)}), \quad \text{where} \quad h(x) = x - 2 + \frac{x}{x-1}. \tag{3}$$

   Analyze the convergence properties of the two methods and determine in particular their order to approximate the roots of $f(x) = 0$.

(b) Consider the optimization problem

$$\max_{x>0} \left(-x^4 + 3x^2 + 2\right). \tag{4}$$

i. (2 points) Verify that problem (4) has a real solution.

ii. (3 points) Write down the Newton's iterate for problem (4). Does Newton's method converges for any initial iterate $x_0 > 0$?

5. The closed Simpson's rule is obtained by using the Lagrange interpolation polynomial of degree 2 at the nodes $x_i = a, c = (a+b)/2, b$, where a and b are the integration limits.

(a) (4 points) Derive the Simpson's rule

$$I_2(f) = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

(b) (6 points) Find the error expression. You may want to use the following result (do not need to prove)

$$f[x_0, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

for some ξ in the smallest interval containing the nodes $x_0, x_1, \ldots, x_n$ and x.