University of Puerto Rico
College of Natural Sciences
Department of Mathematics
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Qualifying Examination

Area: Computational Analysis **Date:17 February 2012**

Solve any three of the following five problems.

1. A matrix A is symmetric diagonally dominant (SDD) if:
(i) for all i, j $A_{ij} = A_{ji}$, (ii) for all i, $A_{ii} \geq \sum_{j \neq i} |A_{ij}|$.

   (a) (1 pt.) Give a 3×3 SDD matrix.

   (b) (2.5 pts.) State Jacobi's iteration for the matrix you gave above. Does it converge?

   (c) (6.5 pts.) Does the Jacobi iteration converge for all SDD matrices? Prove your answer.

   Hint: You may want to consider the spectral radius, and an upper bound to it computed through Ghersghorin's theorem.

2. A matrix P is a projection matrix if $P^2 = P$. A matrix R is a reflection matrix if $R^2 = I$.

   (a) (2 pts.) If P is a projection matrix, prove that $I - P$ is also a projection matrix. Also prove that $P(I - P) = 0$.

   (b) (2 pts.) If R is a reflection matrix, prove that $(I - R)/2$ is a projection matrix.

   (c) (3 pts.) What are the two possible eigenvalues of a projection matrix P? What are the two possible eigenvalues of a reflection matrix R? Prove your answer. **Hint:** Consider the eigenvalue decomposition of the two matrices.

   (d) (3 pts.) Give a simple algorithm for solving linear systems $Px = b$, where P is a projection matrix. The algorithm should use only matrix-vector multiplications and vector comparisons. In your answer, ignore floating point errors. Justify the algorithm.

3. Consider solving the linear system $Ax = b$, where A is an $n \times n$ and nonsingular matrix. Define the condition number of the matrix A as $\text{cond}(A) = \|A\|\|A^{-1}\|$, where $\|\cdot\|$ is an induced matrix norm.

   (a) (3 pts.) Assume that $x \in \mathbb{R}^n$ is the solution of the linear system where $b \neq 0$ and $\delta x \in \mathbb{R}^n$ satisfies
$$A(x + \delta x) = b + \delta b$$
for $\delta b \in \mathbb{R}^n$. Find upper and lower bounds for
$$\frac{\|\delta x\|}{\|x\|} \div \frac{\|\delta b\|}{\|b\|}$$
in terms of the $\text{cond}(A)$.

(b) (2.5 pts.) Show that $\text{cond}(A)_* \leq \text{cond}(A)$, where $\text{cond}(A)_* = \max_i |\lambda_i| / \min_i |\lambda_i|$ and λ_i is eigenvalue of A.

(c) (2.5 pts.) Show that $\text{cond}(A) \geq 1$ and provide an example where $\text{cond}(A)_* = 1$ but $\text{cond}(A)_\infty$ can be made arbitrary large, where $\text{cond}(A)_\infty$ is the condition number of A in the maximum norm $\||\cdot\|_\infty$.

(d) (2 pts.) Assume that the matrix A can be written as $A = UDV^{\mathrm{T}}$, where U and V are orthogonal matrices, $D = \text{diag}(d_1, d_2, \ldots, d_n)$, and $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. Compute $\text{cond}(A)_2$ in terms of the quantities d_i, where $\text{cond}(A)_2$ is the condition number of A in the 2-norm.

4. A function $f(x) \in C^{m+1}([a,b])$ has a root $\alpha \in [a,b]$ of multiplicity $m > 1$ if there is a function $h \in C^{m+1}([a,b])$ such that $h(\alpha) \neq 0$ and

$$f(x) = (x - \alpha)^m h(x). \tag{1}$$

(a) (3 pts.) Show that if $f(x)$ has a root α of multiplicity m, then the Newton's method converges linearly to α with rate of convergence $(m-1)/m$.

(b) (3 pts.) Prove that for a function $f(x)$ with a root α of multiplicity m the modified Newton's method

$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

converges locally and quadratically to α.

(c) Apply the Newton's method and the modified Newton's method to find the root of the following function

$$f(x) = x^m, \quad m > 1.$$

(i) (2 pts.) Estimate the error of each method.

(ii) (2 pts.) Find the value of m that makes the Newton's method to converge slower than the bisection method.

5. Consider the initial value problem

$$y'(x) = f(x, y(x)), \quad a \leq x \leq b, \quad y(a) = y_0, \tag{2}$$

and the linear multi-step method to approximate its solution

$$\begin{aligned} y_{n+k} &= \alpha_{k-1} y_{n+k-1} + \alpha_{k-2} y_{n+k-2} + \ldots + \alpha_0 y_n \\ &\quad + h\left[\beta_k f(x_{n+k}, y_{n+k}) + \beta_{k-1} f(x_{n+k-1}, y_{n+k-1}) + \ldots + \beta_0 f(x_n, y_n)\right], \end{aligned} \tag{3}$$

where the starting points $y_0, y_1, \ldots, y_{k-1}$ are given.

(a) (4.5 pts.) Find conditions on the α's and β's in (3) that lead to an implicit linear one-step method which is stable and at least of order two.

(b) (2.5 pts.) This is an implicit method and to approximate y_{k+1} (when it is needed to evaluate $f(x_{k+1}, y_{k+1})$), in each iteration, let's consider an explicit method as a predictor. Write down the resultant methods if we approximate y_{k+1}, in each iteration, by using the Euler's method.

(c) (3 pts.) Show that the resultant method in (b) is at least of order two.