University of Puerto Rico
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Qualifying Examination

Area: Computational Analysis Date: 7 March 2011

Solve any three of the following five problems.

1. Let A be an $n \times n$ real symmetric positive definite matrix.

   (a) (50%) (*Cholesky decomposition.*) Show that there is a unique $n \times n$ real lower triangular matrix L with $l_{ii} > 0$, $i = 1, 2, \ldots, n$, satisfying $A = LL^{\mathrm{T}}$.

   (b) (50%) Using the Cholesky method calculate the decomposition $A = LL^{\mathrm{T}}$ for

$$A = \begin{bmatrix} 2.25 & -3.00 & 4.50 \\ -3.00 & 5.00 & -10.0 \\ 4.50 & -10.0 & 34.0 \end{bmatrix}.$$

2. Consider the linear system

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

   where A is an $n \times n$ nonsingular matrix. If the system is written into the form

$$\mathbf{x} = H\mathbf{x} + \mathbf{c} \tag{2}$$

   then one can approximate the solution of (1) by using the iteration

$$\mathbf{x}^{(k+1)} = H\mathbf{x}^{(k)} + \mathbf{c}, \tag{3}$$

   with an arbitrary $\mathbf{x}^{(0)}$.

   (a) (35%) Let $\mathbf{x}^*$ be the unique solution of (2), prove that if $\rho(H) < 1$, then for any $\mathbf{x}^{(0)}$ the iterates (3) converge to $\mathbf{x}^*$. (Here $\rho(H)$ is the spectral radius of H.)

   (b) (35%) Define H and $\mathbf{c}$ for the Jacobi and Gauss-Seidel methods. Find conditions on A for the methods to be well defined.

   (c) (30%)For the matrix

$$A = \begin{bmatrix} 1 & 1/4 & 1/2 \\ 1/4 & 1 & 1/4 \\ 1/4 & 1/4 & 1 \end{bmatrix}.$$

   Which method converges faster, Jacobi or Gauss-Seidel?

3. (a) (30%) Let A be a real symmetric $n \times n$ matrix, show that

$$\operatorname{cond}_2(A) = \frac{\max_{1 \le i \le n} |\lambda_i|}{\min_{1 \le i \le n} |\lambda_i|},$$

where $\{\lambda_i\}_{i=1}^n$ is the set of eigenvalues of A.

(b) (30%) Show that the eigenvalues of a real symetric matrix are real.

(c) (40%) By using the Gershgorin theorem, compute an upper bound for $\operatorname{cond}_2(A)$, where A is given by

$$A = \begin{bmatrix} 5.2 & 0.6 & 2.2 \\ 0.6 & 6.4 & 0.5 \\ 2.2 & 0.5 & 4.7 \end{bmatrix}.$$

4. To solve the nonlinear system

$$\begin{aligned} x_1 &= -\frac{1}{81}\cos x_1 + \frac{1}{9}x_2^2 + \frac{1}{3}\sin x_3, \\ x_2 &= \frac{1}{3}\sin x_1 + \frac{1}{3}\cos x_3, \\ x_3 &= -\frac{1}{9}\cos x_1 + \frac{1}{3}x_2 + \frac{1}{6}\sin x_3, \end{aligned} \tag{4}$$

consider the fixed-point iteration

$$\mathbf{x}^{(k+1)} = \Psi(\mathbf{x}^{(k)}), \tag{5}$$

where $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ and $\Psi(\mathbf{x})$ is the right-hand side of the system in (4).

(a) (20%) Do two iterations of the fixed point (5) starting with $\mathbf{x^{(0)}} = [\mathbf{0}, \mathbf{0}, \mathbf{0}]^{\mathrm{T}}$.

(b) (40%) Analyze the convergence of the iteration to compute the fixed point $\mathbf{x}^* = [0, 1/3, 0]^{\mathrm{T}}$.

(c) (40%) Rewrite the nonlinear system (4) as

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}$$

and write down the Newton's method for the resultant system.

5. For the differential equation

$$y'(t) = f(t, y(t)), \quad y(0) = y_0, \tag{6}$$

consider the following method

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}\left(4f(t_{n+1}, y_{n+1}) - f(t_n, y_n) + 3f(t_{n-1}, y_{n-1})\right), \tag{7}$$

where $t_n = nh$.

(a) (40%) Find the local truncation error of this method.

(b) (20%) Is the method (7) consistent with the differential equation in (6)?

(c) (40%) Determine the stability of this method.