University of Puerto Rico
College of Natural Sciences
Department of Mathematics
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MS Qualifying Examination

Area: Computational Analysis **Date: 24 August 2018**

Solve three of the following problems. If more than three problems are answered, please indicate the three problems that must be graded.

1. Consider the following approach to developing a compact method for producing the LU factorization of a nonsingular $n \times n$ matrix A. Write

$$A = \begin{bmatrix} \hat{A} & \mathbf{d} \\ \mathbf{c}^{\mathrm{T}} & \alpha \end{bmatrix} \tag{1}$$

where $\hat{A}$ is $n-1 \times n-1$, $\mathbf{c}$ and $\mathbf{d} \in \mathbb{R}^{n-1}$, and $\alpha \in \mathbb{R}$. As a step in an induction process, assume $\hat{A}$ is nonsingular and the LU decomposition is known

$$\hat{A} = \hat{L}\hat{U},$$

where $\hat{L}$ is lower triangular with ones on the diagonal and $\hat{U}$ is upper triangular with nonzero elements on the diagonal. Then seek $A = LU$ in the form

$$A = \begin{bmatrix} \hat{L} & \mathbf{0} \\ \mathbf{m}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \hat{U} & \mathbf{q} \\ \mathbf{0}^{\mathrm{T}} & \gamma \end{bmatrix},$$

where $\mathbf{m}$ and $\mathbf{q} \in \mathbb{R}^{n-1}$, $\mathbf{0}$ is the zero vector of length $n-1$, and $\gamma \in \mathbb{R}$.

   (a) (5 points) Show that $\mathbf{m}$, $\mathbf{q}$, and γ can be found, and describe how to do so.

   (b) (4 points) Apply the method obtained in (a) to find the LU decomposition of the following matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}.$$

   (c) (1 point) However, one must be careful with this method because in the partition of A in (1) even when A is nonsingular, $\hat{A}$ could be singular, provide an example of such a case.

2. Let A be an $m \times n$ real matrix and consider the QR factorization of A, where Q is an $m \times m$ orthogonal matrix and R is an $m \times n$ upper trapezoidal matrix. Let's assume that $m \geq n$ and A is a full rank matrix.

(a) (3 points) By writing

$$R = \begin{bmatrix} \tilde{R} \\ O \end{bmatrix}$$

where $\tilde{R}$ is an $n \times n$ upper triangular matrix and O is the $(m-n) \times n$ zero matrix. Prove that the solution of the linear system $A\mathbf{x} = \mathbf{b}$ in the least-squares sense can be obtained by solving

$$\tilde{R}\mathbf{x} = \mathbf{c}$$

where $\mathbf{c}$ is the vector of the first n elements of $Q^{\mathrm{T}}\mathbf{b}$. Further, show that the square error is given by the square of the two-vector norm $(\|\cdot\|_2)$ of the vector that contains the rest $m-n$ elements of $Q^{\mathrm{T}}\mathbf{b}$.

(b) (3 points) Prove that the singular values σ_i of A are real and nonnegative. Further, show that $\max_{1 \le i \le n} \sigma_i = \|A\|_2$.

Now, consider the following matrix and column vector

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

(c) (2 points) Verify that the following matrix is orthogonal

$$Q = \begin{bmatrix} -\sqrt{2/3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

and find the upper trapezoidal matrix R that with Q produces the QR decomposition of A.

(d) (2 points) Using the QR factorization from problem (2c), and the method from problem (2a) solve the linear system $A\mathbf{x} = \mathbf{b}$ in the least-squares sense and report the square of the residual norm.

3. Let A be a nonsingular $n \times n$ matrix and $\kappa(A) = \|A\|\,\|A^{-1}\|$ its condition number.

(a) (2 points) Show that $\kappa(\alpha A) = \kappa(A)$ for any nonzero scalar α.

(b) (5 points) Show that $\kappa_2(A) = \kappa_2(QA) = \kappa_2(AQ)$ for any orthogonal $n \times n$ matrix. **Note:** $\kappa_2(A)$ is the condition number of A with respect to the two–matrix norm $\|\cdot\|_2$.

(c) (3 points) Consider the linear system

$$\begin{cases} 0.001x_1 &= b_1, \\ x_1 + 10000x_2 &= b_2. \end{cases}$$

where b_1, b_2 are arbitrary. Show that the condition number of the coefficient matrix is at least 10^7. However, give a geometrical argument to the fact that the coefficient matrix can still be considered well conditioned.

4. Let A be a real $n \times n$ matrix and assume that $\lambda \in \mathbb{C}$ is a simple eigenvalue of A with corresponding eigenvector $\mathbf{v}$ such that $\mathbf{v}^{\mathrm{T}}\mathbf{v} = 1$. The pair $(\mathbf{v}, \lambda)$ is a solution of $\mathbf{F}(\mathbf{x}, \gamma) = \mathbf{0}$ where $\mathbf{F} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is given by

$$\mathbf{F} = \begin{bmatrix} A\mathbf{x} - \gamma\mathbf{x} \\ \mathbf{x}^{\mathrm{T}}\mathbf{x} - 1 \end{bmatrix}.$$

   (a) (4 points) Write down the equations for Newton's method for approximating a root or solution of $\mathbf{F}(\mathbf{x}, \gamma) = \mathbf{0}$.

   (b) (4 points) Show that the derivative of $\mathbf{F}$ at $(\mathbf{v}, \lambda)$ is nonsingular. **Note:** You may assume without proof that $\mathrm{Range}(A - \lambda I) \cap N(A - \lambda I) = \{\mathbf{0}\}$.

   (c) (2 points) What can you conclude about the convergence (global or local and order) of Newton's method to the solution $(\mathbf{v}, \lambda)$ of $\mathbf{F}(\mathbf{x}, \gamma) = \mathbf{0}$.

5. Consider the open Newton-Cotes rule $I_1^{\mathrm{o}}(f)$ that is obtained by using the Lagrange interpolating polynomial of degree 1.

   (a) (3 points) Derive the rule $I_1^{\mathrm{o}}(f)$.

   (b) (3 points) Find the error expression $E_1^{\mathrm{o}}(f)$ for the rule $I_1^{\mathrm{o}}(f)$.

   (c) (2 points) Using the rule $I_1^{\mathrm{o}}(f)$, derive a composite method $I_{1,m}^{\mathrm{o}}(f)$ that includes the error term.

   (d) (2 points) Find the number of subintervals m that is needed for the composite rule $I_{1,m}^{\mathrm{o}}(f)$ to compute the value of the integral

$$I(f) = \int_{-1}^{1} \frac{dx}{1 + x^2}$$

   to 8 digits of accuracy.