University of Puerto Rico
College of Natural Sciences
Department of Mathematics
Río Piedras Campus

Qualifying Examination

Area: Computational Analysis **Date: 4 September 2015**

Solve any three of the following five problems.

1. Let $\hat{A}$ be an $(n-1)\times(n-1)$ tridiagonal and diagonally dominant matrix. Let A be an $n\times n$ matrix written as
$$A = \begin{bmatrix} \hat{A} & \mathbf{a} \\ \mathbf{b}^{\mathrm{T}} & \alpha \end{bmatrix},$$
where $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^{n-1}$ and $0 \neq \alpha \in \mathbb{R}$.

   (a) (5 points) Develop a method for producing the LU factorization of A, where
$$L = \begin{bmatrix} \hat{L} & \mathbf{0} \\ \mathbf{c}^{\mathrm{T}} & \lambda \end{bmatrix}, \qquad U = \begin{bmatrix} \hat{U} & \mathbf{d} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix},$$
$\hat{L}$ is an $(n-1)\times(n-1)$ lower triangular matrix, $\hat{U}$ is an $(n-1)\times(n-1)$ upper triangular matrix with ones on the diagonal, $\mathbf{c}$ and $\mathbf{d} \in \mathbb{R}^{n-1}$, and $0 \neq \lambda \in \mathbb{R}$.

   (b) (2 points) Is A nonsingular?

   (c) (3 points) Apply the method obtained in (a) to find the LU decomposition of the following matrix
$$A = \begin{bmatrix} 4 & -1 & 0 & 1 \\ 1 & 4 & -1 & 1 \\ 0 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

2. For the matrix
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$
and an arbitrary vector $\mathbf{b}$:

   (a) (2 point) Write the Gauss-Jacobi method as $\mathbf{x}^{(k+1)} = H\mathbf{x}^{(k)} + \mathbf{c}$ and identify the matrix H and the vector $\mathbf{c}$.

   (b) (4 point) Prove that the Gauss-Jacobi method converges for this matrix. *Hint*: You might consider finding an upper bound for the spectral radius of H and then prove that that bound is not the maximum.

   (c) (4 point) A is a symmetric diagonally dominant (SDD) matrix. Does Gauss-Jacobi converge for all SDD matrices? Prove your answer.

3. Consider the function
$$f(x) = -x^4 + 3x^2 + 2.$$
   (a) (2 point) Verify that this function has real roots.
   (b) (4 points) Apply Newton's method to $f(x)$ with starting point $x_0 = 1$ and explain why the method fails to converge.
   (c) (4 point) Find a starting point for which the method converges. Prove that the convergence of this method is order two by computing
$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2}.$$

4. For the function $f(x) = 1 - x^n$, for n even, apply the following formulas to approximate $\int_{-1}^{1} f(x)dx$ (do not use composite rule over the stated interval):
   (a) (2 points) The trapezoidal formula over the interval $[-1, 1]$.
   (b) (2 points) The trapezoidal formula over the interval $[-1, 0]$ and then over the interval $[0, 1]$.
   (c) (2 points) Simpson formula over the interval $[-1, 1]$.
   (d) (4 points) Explain the result in each case.

5. Consider the following integration method
$$u_{i+2} = u_i + \frac{h}{3}\left(f(u_i) + 4f(u_{i+1}) + f(u_{i+2})\right).$$
   (a) (2 point) Is this method consistent with the differential equation $y'(x) = f(x, y(x))$?
   (b) (3 points) Determine the stability of this method.
   (c) (5 points) Find the leading term in the local truncation error of this method and the exact order of convergence.