University of Puerto Rico
College of Natural Sciences
Department of Mathematics
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Qualifying Examination

Area: Computational Analysis **Date: 24 September 2012**

Solve any three of the following five problems.

1. Let

$$A = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

   (a) (3 points) Find the Cholesky factorization of the matrix A.

   (b) (3 points) Find the Cholesky factorization of the matrix $B = P^T AP$, where P is the permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

   (c) (2 point) Instead of solving the linear system $Ax = b$, we can solve the linear system $(P^T AP)P^T x = P^T b$. Prove this claim.

   (d) (2 points) Part (c) implies that we can solve $By = P^T b$, where $y = P^T x$. How can we recover x from y in a fast way, given that P is a permutation matrix?

2. Given the matrix

$$A = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix},$$

   for $\gamma \geq 0$.

   (a) (2 points) Compute $\mathrm{Cond}_\infty(A)$.

   (b) (5 points) Consider the linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is such that $\mathbf{x} = [1 - \gamma, 1]^T$ is the solution. Find lower and upper bounds for $\|\delta\mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty$ in terms of $\|\delta\mathbf{b}\|_\infty / \|\mathbf{b}\|_\infty$ when $\delta\mathbf{b} = (\delta_1, \delta_2)^T$.

   (c) (3 points) Is the problem well- or ill-conditioned?

3. Consider the matrix A given by

$$A = \begin{bmatrix} 0.1 & 0 & 1.0 & 0.2 \\ 0.2 & 0.3 & 0.1 & 0.3 \\ 0.1 & 0.1 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \end{bmatrix},$$

(a) (2 points) Verify that $\|A\|_\infty > 1$.

(b) (5 points) Show, however, that $\rho(A) < 1$ by considering DAD^{-1} for a suitable diagonal D and applying the Gerschgorin's theorem. (Here $\rho(A)$ is the spectral radius of A, i.e., $\max_i |\lambda_i|$, where λ_i is eigenvalue of A.)

(c) (3 points) Apply three iterations of the Power method to approximate the eigenvector associated with the dominant eigenvalue of A, compute also the eigenvalue.

4. Consider the following fixed-point problem $g(x) = wx + \dfrac{(1-w)A}{x^2}$.

(a) (3 points) Find the values of w for which the fixed-point iteration

$$x_{n+1} = wx_n + \frac{(1-w)A}{x_n^2}$$

will converge to $A^{1/3}$ (provided x_0 is chosen sufficiently close to $A^{1/3}$).

(b) (2 points) For what value of w will the convergence be quadratic?

(c) (5 points) For the case in (b) calculate

$$\lim_{n\to\infty} \frac{x_{n+1} - A^{1/3}}{(x_n - A^{1/3})^2},$$

assuming x_0 has been chosen sufficiently close to $A^{1/3}$.

5. Consider the linear multistep method

$$u_n = u_{n-4} + \frac{4h}{3}[2f_{n-1} - f_{n-2} + 2f_{n-3}]$$

for solving the ODE $y'(x) = f(x, y(x))$.

(a) (3 points) Is the method consistent?

(b) (5 points) Analyze the stability of the method.

(c) (2 points) Does the method converge? Explain.