University of Puerto Rico
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Qualifying Examination

Area: Computational Analysis Date: 1 December 2004

Solve any three of the following five problems.

1. Consider the linear system

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where A is a real $n \times n$ nonsingular matrix. Let A be written in the form

$$A = M - N,$$

where M is nonsingular. The equation (1) can be written in the form

$$\mathbf{x} = M^{-1}\mathbf{b} + M^{-1}N\mathbf{x}.$$

This expression suggests an iterative scheme of the form

$$\mathbf{x}^{k+1} = M^{-1}\mathbf{b} + M^{-1}N\mathbf{x}^{k}, \quad k = 0, 1, 2, \ldots, \tag{2}$$

with an arbitrary $\mathbf{x}^0$.

(a) (30%) Let $\mathbf{x}^*$ be the unique solution of (1), prove that if $\rho(M^{-1}N) < 1$, then for any $\mathbf{x}^0$ the iterates (2) converge to $\mathbf{x}^*$. (Here $\rho(H)$ is the spectral radius of H.)

(b) (30%) Define M and N for the Jacobi and Gauss-Seidel methods. Find conditions on A for the methods to be well defined.

(c) (40%) Show that the Jacobi method is covergent for all matrices A with

$$|a_{ii}| > \sum_{k \neq i} |a_{ik}|, \quad \text{for} \quad i = 1, 2, \ldots, n.$$

2. Let $A\mathbf{x} = \mathbf{b}$ be given with

$$A = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}.$$

The exact solution is $\mathbf{x}^T = (1, -1)$. Further, let two approximate solutions

$$\mathbf{x}_1^T = (0.999, -1.001), \quad \mathbf{x}_2^T = (0.341, -0.087)$$

be given.

(a) (30%) Compute the residuals $\mathbf{r}(\mathbf{x}_1), \mathbf{r}(\mathbf{x}_2)$. Does the more accurate solution have a smaller residual?

(b) (30%) Determine $\text{cond}_\infty(A)$ (with respect to the maximum norm) given

$$A^{-1} = \begin{bmatrix} 659000 & -563000 \\ -913000 & 780000 \end{bmatrix}.$$

(c) (40%) Express $\mathbf{x}_i - \mathbf{x} = \Delta\mathbf{x}_i$ using the residual for $\mathbf{x}_i$, $i = 1, 2$. Does this provide an explanation for the discrepancy observed in 2a? Explain.

3. (a) (30%) Let A be a real symmetric $n \times n$ matrix, show that

$$\text{cond}_2(A) = \frac{\max_{1 \le i \le n} |\lambda_i|}{\min_{1 \le i \le n} |\lambda_i|},$$

where $\{\lambda_i\}_{i=1}^n$ is the set of eigenvalues of A.

(b) (30%) Show that the eigenvalues of a real symetric matrix are real.

(c) (40%) By using the Gershgorin theorem, compute an upper bound for $\text{cond}_2(A)$, where A is given by

$$A = \begin{bmatrix} 5.2 & 0.6 & 2.2 \\ 0.6 & 6.4 & 0.5 \\ 2.2 & 0.5 & 4.7 \end{bmatrix}.$$

4. Assume that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, and let α be a solution of $f(x) = 0$, with $f'(\alpha) \neq 0$. The Steffensen's method

$$x_{k+1} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}$$

is a quasi-Newton method.

   (a) (50%) Show that the method has the form

   $$x_{k+1} = x_k - q(x_k) f(x_k).$$

   Give $q(x)$, and show that there is a constant c such that

   $$|q(x) - [f'(x)]^{-1}| \leq c|f(x)|.$$

   (b) (50%) Show that the Steffensen's method is locally and quadratically convergent to α.

   Hint: Write $f(x) = (x - \alpha)h(x)$ with $h(\alpha) \neq 0$, and visualize the Steffensen's method as a fixed point iteration.

5. Consider the general linear multistep method

$$y_{n+k} = -\alpha_{k-1} y_{n+k-1} - \alpha_{k-2} y_{n+k-2} - \ldots - \alpha_0 y_n + h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{3}$$

where a set of starting points $y_0, y_1, \ldots, y_{k-1}$ is provided.

   (a) (60%) Find the most accurate implicit linear two-step method.

   (b) (40%) Find also the first term in the local truncation error.