University of Puerto Rico
College of Natural Sciences
Department of Mathematics
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MS Qualifying Examination

Area: Computational Analysis **Date: 5 February 2020**

Solve three of the following problems. If more than three problems are answered, please indicate the three problems that must be graded.

1. Consider the iterative method

$$\mathbf{x}^{(k+1)} = H\mathbf{x}^{(k)} + \mathbf{c} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^n$, H is an $n \times n$ real matrix, and $\mathbf{x}^{(0)} \in \mathbb{R}^n$ is a given initial guess. Assume that $\|H\| < 1$, where $\|\cdot\|$ is a matrix norm induced by the vector norm. Do the following:

(a) (4 points) Prove that the iterative method (1) converges to the unique solution of the linear system

$$\mathbf{x} = H\mathbf{x} + \mathbf{c}.$$

(b) (3 points) Prove the following error bound

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \leq \| (I - H)^{-1} \| \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|.$$

(c) (3 points) In addition, from the iteration in (1) and the result in (b), prove that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \leq \|H\|^k \|\mathbf{x}^{(0)}\| + \frac{\|H\|^k \|\mathbf{c}\|}{I - \|H\|}.$$

2. For the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

(a) (2 points) Show that the eigenvalues of A are real.

(b) (3 points) By using the Gershgorin's Theorem locate the eigenvalues of A in real intervals.

(c) (2 points) Use the result in (b) to prove that A in nonsingular.

(d) (3 points) Use the result in (b) to compute an upper bound for the condition number of A, $\text{cond}_2(A)$, where $\text{cond}_2(A) = \|A\|_2 \|A^{-1}\|_2$ and $\|\cdot\|_2$ is the two-matrix norm. Give also a lower bound for $\text{cond}_2(A)$.

3. Let us assume that $f \in \mathbb{R} \to \mathbb{R}$ is at least twice continuously differentiable, that $f(\alpha) = 0$, and $f'(\alpha) \neq 0$. Newton's method for approximating the root of $f(x) = 0$ has the following iterative formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \qquad i = 0, 1, 2, \ldots, \tag{2}$$

where x_0 is a given initial value.

(a) (3 points) Show that if x_0 is chosen sufficiently close to α, then the Newton's iterates (2) will converge to α.

(b) (3.5 points) Assume that α is a simple root of $f(x) = 0$ and show that

$$\lim_{i \to \infty} \frac{|\alpha - x_{i+1}|}{|\alpha - x_i|^2} = \frac{|f''(\alpha)|}{2|f'(\alpha)|},$$

which proves that the iterates have order of convergence two.

(c) (3.5 points) Now, assume that α is a multiple root of $f(x) = 0$ with multiplicity m ($m \geq 2$) and prove that Newton's method is linearly convergent. Find the rate of convergence of Newton's method for this case.

4. A close Newton-Cotes rule for approximating the finite integral of a function $f(x)$ is obtained as follows:

$$I_n(f) = \int_a^b p_n(x)dx,$$

where $p_n(x)$ is the Lagrange interpolating polynomial of $f(x)$ at the nodes $a = x_0$, x_1, x_2, ..., $x_n = b$,

$$p_n(x) = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

(a) (3 points) Show that

$$I_1(f) = \frac{b-a}{2}\left(f(a) + f(b)\right),$$

also known as the trapezoidal rule.

(b) (3 points) Show that the error of the rule $I_1(f)$ is given by

$$E_1(f) = -\frac{f''(\eta)}{12}(b-a)^3,$$

for some $\eta \in (a, b)$. Recall that the interpolation error at the point x is given by

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

where ξ is in the smallest interval containing $x, x_0, x_1, \ldots, x_n$.

(c) (2 points) Find the composite trapezoidal rule $I_{1,m}(f)$ that includes the error term.

(d) (2 points) Find the number of subintervals m that is needed for the composite trapezoidal rule to compute the value of the integral

$$I(f) = \int_1^2 x\ln(x)dx$$

to 4 digits of accuracy.

5. Consider the initial value problem

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(0) &= y_0 \end{aligned} \tag{3}$$

and the numerical method for approximating the solution of (3)

$$u_{i+2} = u_{i+1} + \frac{h}{12}\left(5f(x_{i+2}, u_{i+2}) + 8f(x_{i+1}, u_{i+1}) - f(x_i, u_i)\right), \tag{4}$$

where h is the step length.

(a) (2 point) Is method (4) consistent with the differential equation in (3)?

(b) (2 points) Is method (4) stable?

(c) (6 points) Find the leading term in the local truncation error of method (4) and the order of convergence.