University of Puerto Rico
College of Natural Sciences
Department of Mathematics
Río Piedras Campus

MS Qualifying Examination

Area: Computational Analysis **Date: 11 August 2022**

Solve three of the following problems. Please indicate below which problems you have attempted by circling the appropriate numbers:

1 **2** **3** **4** **5**

1. Consider the matrix
$$A = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix},$$
where $\gamma \geq 0$, and the linear system $A\mathbf{x} = \mathbf{b}$, for some vector $\mathbf{b} \in \mathbb{R}^2$.

   (a) (3 points) Compute
$$\text{Cond}(A)_\infty = \|A\|_\infty \|A^{-1}\|_\infty,$$
where $\|A\|_\infty$ is the maximum-matrix norm of A.

   (b) (3 points) For what values of γ the solution of the linear system by using Gaussian elimination in a machine with double precision might not guarantee any digits of accuracy?

   (c) (4 points) Now, for a value of γ within the range from part (b) solve the linear system with an arbitrary $\mathbf{b} \in \mathbb{R}^2$ using Gaussian elimination and explain why A can still be considered well conditioned.

2. Consider the $n \times n$ Householder matrix H applied to a vector $\mathbf{a} \in \mathbb{R}^n$, where $H = I - 2\mathbf{w}\mathbf{w}^{\mathrm{T}}$ for some unit vector $\mathbf{w}$, such that $H\mathbf{a} = \beta\mathbf{e}_1$, where $\mathbf{e}_1 = [1, 0, 0, \ldots, 0]^{\mathrm{T}}$.

   (a) (3 points) If H is real, show that $\beta = \pm\|\mathbf{a}\|_2$ and explain how to choose the sign.

   (b) (4 points) Given $\mathbf{a}$ and β, explain how to construct $\mathbf{w}$.

   (c) (3 points) Construct the Householder matrix to transform the vector
$$\mathbf{a} = [1, 2, 2]^{\mathrm{T}}.$$

3. Let $g(x)$ be a continuously differentiable function with a fixed point at $x = \alpha$. Check the following properties:

   (a) (3.5 points) If $0 < g'(\alpha) < 1$, then the convergence is monotone, that is, the error $x_k - \alpha$ maintains a constant sign as k increases.

   (b) (3.5 points) If $-1 < g'(\alpha) < 0$, then the convergence is oscillatory, that is, the error $x_k - \alpha$ alternates sign as k increases.

(c) (3 points) If $|g'(\alpha)| > 1$, then the iterates diverge. More precisely, if $g'(\alpha) > 1$, the sequence is monotonically diverging, while for $g'(\alpha) < -1$, it diverges with oscillatory sign.

4. Simpson's rule approximates the integral $I(f) = \int_a^b f(x)dx$ by using the Lagrange interpolating polynomial at the points $(a, f(a)), (b, f(b))$, and $(c = (a+b)/2, f(c))$.

   (a) (5 points) Show that Simpson's rule is

   $$I_2(f) = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

   (b) (5 points) Prove that the error expression $E_2(f)$ for Simpson's rule is given by

   $$E_2(f) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi),$$

   for some $\xi \in (a, b)$.

5. For the initial value problem

   $$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0 \end{aligned} \tag{1}$$

   consider the methods

   $$y_{i+1} = y_i + h\left[\alpha f(x_i, y_i) + (1-\alpha) f(x_{i-1}, y_{i-1})\right], \tag{2}$$

   where $\alpha \in \mathbb{R}$, $y_i = y(x_i)$, and h is the step length.

   (a) (5 points) Find the leading term in the local truncation error of method (2) and determine the order of accuracy of the method as a function of α.

   (b) (5 points) For the special case $f(x, y) = \lambda y$ with $\lambda \in \mathbb{R}$ find the maximum h such that the method is stable for $\alpha = 1$. Note that h may depend on λ.