Minimum Rank of Subgraphs of Hypercubes

Héctor D. Torres Aponte, María Ñeco Arroyo

University of Puerto Rico – Río Piedras Campus

July 28, 2010

Matrices and Graphs

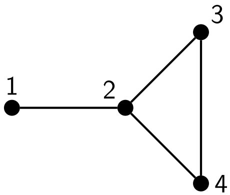
Definition

The graph $\mathcal{G}(A) = (V, E)$ of $n \times n$ matrix A is a graph where:

1. $V = \{1, ..., n\}$
2. $E = \{ij : a_{ij} \neq 0\}$
3. Diagonal of A is ignored

Example:

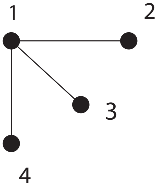
$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 2 & -1 & -5 \\ 0 & 3 & -5 & 2 \end{bmatrix}$$



Definition

The set of symmetric matrices described by a graph G (over $\mathbb{R}$) is
$\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$

Example:



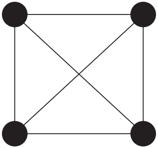
$$\begin{bmatrix} ? & a & b & c \\ a & ? & 0 & 0 \\ b & 0 & ? & 0 \\ c & 0 & 0 & ? \end{bmatrix}$$

Minimum Rank

Definition

The minimum rank of G is $mr(G) = \min\{\text{rank}(A) \mid A \in \mathcal{S}(G)\}$.

Example:



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

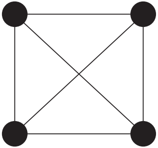
$$mr(K_4) = 1$$

Maximum Nullity

Definition

The *maximum nullity of G* *is* $M(G) = \max\{corank(A) : A \in \mathcal{S}(G)\}$

Example:



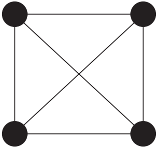
1. $M(K_4) = 3$

Maximum Nullity

Definition

The *maximum nullity of* G *is* $M(G) = \max\{corank(A) : A \in \mathcal{S}(G)\}$

Example:



1. $M(K_4) = 3$
2. $mr(K_4) = 1$
3. $|K_4| = 4$

Note that $M(K_4) = |K_4| - mr(K_4)$

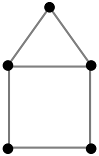
Induced Subgraph

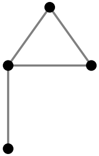
Definition

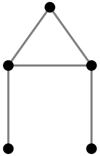
A graph $G' = (V', E')$ is a subgraph of graph $G = (V, E)$ if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$.

Example:

$G =$





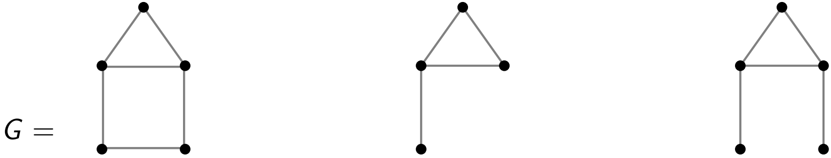


Induced Subgraph

Definition

A graph $G' = (V', E')$ is a subgraph of graph $G = (V, E)$ if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$.

Example:



Definition

A subgraph $G[V']$ is an induced subgraph of G by V' if $\forall v_i, v_j \in V'$, $v_i v_j \in E(G) \Rightarrow v_i v_j \in E(G[V'])$.

Observation

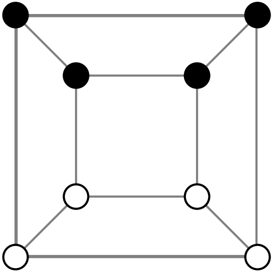
1. *If* G' *is an induced subgraph of* G*, then* $mr(G') \leq mr(G)$.
2. $mr(G) + M(G) = |G|$
3. *If* $G = \cup_{i=1}^{h} G_i$ *then* $\mathrm{mr}(G) \leq \sum_{i=1}^{h} \mathrm{mr}(G_i)$.

Coloring

Color Change Rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

Example:

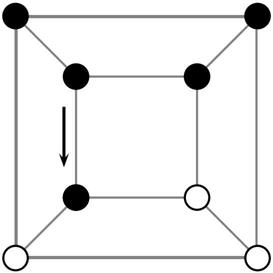


Coloring

Color Change Rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

Example:

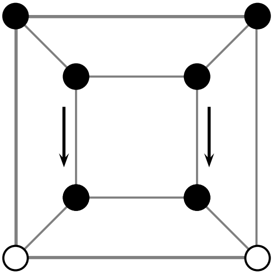


Coloring

Color Change Rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

Example:

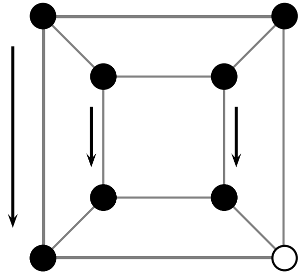


Coloring

Color Change Rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

Example:

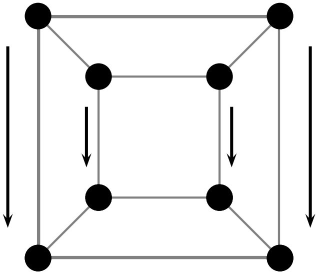


Coloring

Color Change Rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

Example:



Zero Forcing Set and Number

Definition

- *Derived coloring is the result of applying the color-change rule until no more changes are possible.*
- *Zero forcing set (ZFS)* $Z \subseteq V$, *s.t. if initially* $v_i \in Z$ *are colored black and* $v_j \notin Z$ *are colored white, then the derived coloring is all black.*
- *Zero forcing number* $Z(G) = \min\{|Z| : Z \text{ is a ZFS }\}$.

Theorem (AIM08)

For any graph G, $M(G) \leq Z(G)$

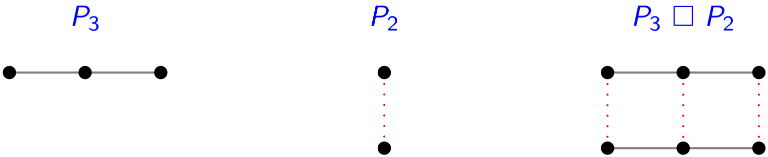
Hypercube

Definition

The Cartesian Product of two graphs G and H, denoted $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if:

1. $u = u'$ *and* $vv' \in E(H)$, *or*
2. $v = v'$ *and* $uu' \in E(G)$

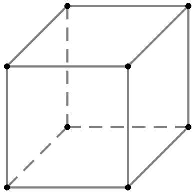
Example: $P_3 \,\square\, P_2$



Definition (Hypercube)

$Q_o = (V, E)$ where $|V| = 1, |E| = 0$
$Q_d = Q_{d-1} \square P_2$

Example: $Q_4 = Q_3 \square P_2$

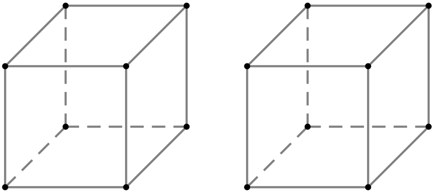


Hypercube

Definition (Hypercube)

$Q_o = (V, E)$ where $|V| = 1, |E| = 0$
$Q_d = Q_{d-1} \,\square\, P_2$

Example: $Q_4 = Q_3 \,\square\, P_2$

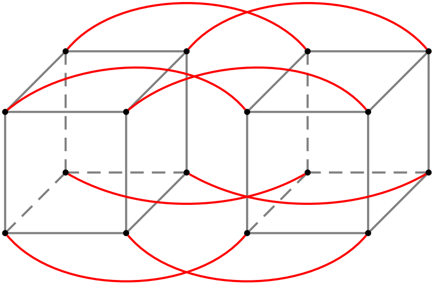


Hypercube

Definition (Hypercube)

$Q_o = (V, E)$ where $|V| = 1, |E| = 0$
$Q_d = Q_{d-1} \square P_2$

Example: $Q_4 = Q_3 \square P_2$

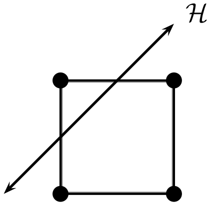


Hypercube

Definition (Cut Complex)

A Cut Complex $\mathcal{C}$ is a subgraph of Q_d for which there is a $(d-1)$-dimensional hyperplane $\mathcal{H}$ that strictly separates the vertices of $\mathcal{C}$ from the rest of the vertices of Q_d.

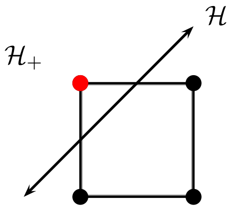
Example:



Definition (Cut Complex)

A Cut Complex $\mathcal{C}$ is a subgraph of Q_d for which there is a $(d-1)$-dimensional hyperplane $\mathcal{H}$ that strictly separates the vertices of $\mathcal{C}$ from the rest of the vertices of Q_d.

Example:

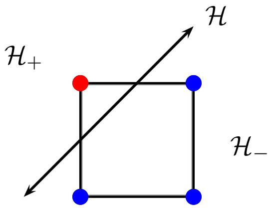


Hypercube

Definition (Cut Complex)

A Cut Complex $\mathcal{C}$ is a subgraph of Q_d for which there is a $(d-1)$-dimensional hyperplane $\mathcal{H}$ that strictly separates the vertices of $\mathcal{C}$ from the rest of the vertices of Q_d.

Example:

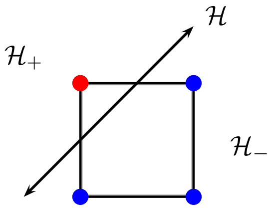


Hypercube

Definition (Cut Complex)

A Cut Complex $\mathcal{C}$ is a subgraph of Q_d for which there is a $(d-1)$-dimensional hyperplane $\mathcal{H}$ that strictly separates the vertices of $\mathcal{C}$ from the rest of the vertices of Q_d.

Example:



1. Red vertices are in $\mathcal{C}_0$.
2. Blue vertices are in $\mathcal{C}_0'$.
3. Cut complexes are induced subgraph of Q_d.

Covering

Definition

The path cover number of G,$P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G; such a set of paths realizing $P(G)$ is called a minimal path cover.

Theorem (Johnson,Leal Duarte 99)

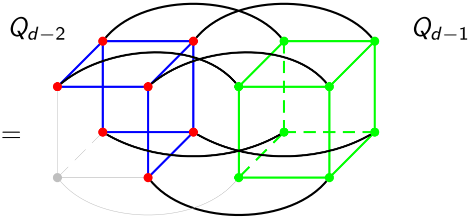
Let T be a tree,
$M(T) = P(T) = |T| - mr(T)$

Main Result

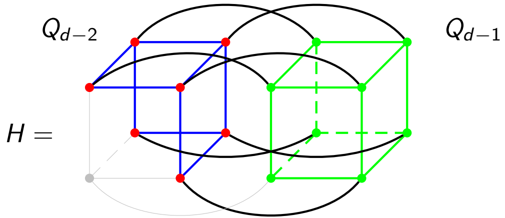
Theorem

Let H be an induced subgraph of $Q_d = Q_{d-1} \square P_2$ such that $Q_{d-1} \subseteq H$ and H contains Q_{d-2} from the other copy of Q_{d-1}. Then $\operatorname{mr}(H) = \operatorname{mr}(Q_d) = 2^{d-1}$ *and* $M(H) = Z(H) = |H| - 2^{d-1}$.

Example:

$H =$ 

Example:



- H is an induced subgraph of Q_4
  - $\text{mr}(H) \leq \text{mr}(Q_4)$
  - $\text{mr}(H) \leq 8$
- Order of graph: $|H| = 15$
- Zero forcing number: $Z(H) \leq 7$
- $\text{M}(H) \leq \text{Z}(H) \Rightarrow \text{M}(H) \leq 7$
- $\text{mr}(H) + \text{M}(H) = |H|$
  - $\Rightarrow \text{mr}(H) = |H| - \text{M}(H)$
  - $\Rightarrow \text{mr}(H) \geq 15 - 7 = 8$
  - $\Rightarrow \text{mr}(H) = 8 = \text{mr}(Q_4)$
- $\text{M}(H) = |H| - \text{mr}(H)$
- $\text{M}(H) = 7 = \text{Z}(H)$

Conjecture

Conjecture

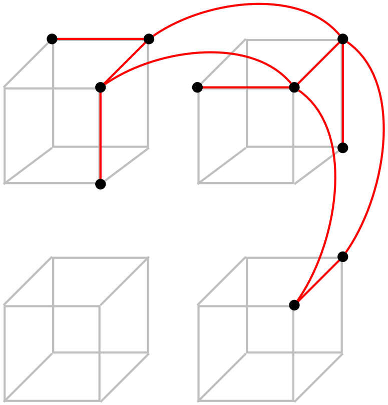
If H is a cut-complex of Q_d then $\mathrm{M}(H) = Z(H)$.

For $d = 1, 2, 3, 4$ and H a *cut-complex* of Q_d; $\mathrm{mr}(H), \mathrm{M}(H), Z(H)$ have been computed.

Remark

Remark

There is an example of an induced subgraph H of Q_5 *with* $\mathrm{M}(H) < \mathrm{Z}(H)$.
H is not a cut-complex.



References

- AIM Minimum Rank-Special Graphs Work Group, Zero forcing set and minimum rank of graphs, *Linear Algebra Appl.* 428(7) (2008) 1628–1648.
- L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, G. Tims. Techniques for determining the minimum rank of a small graphs, *Linear Algebra Appl.* 432(11) (2010) 2995–3001.
- L. DeLoss, J. Grout, T. McKay, J. Smith, G. Tims, Program for calculating bounds on the minimum rank of a graph using Sage, `http:arxiv.org/abs/0812.1616`
- M.R. Emamy-Khansary. On the cuts and cut number of the 4-Cube. *J*ournal of Combinatorical Theory, Series A., 41 (1986)221–227.
- S. Fallat, L. Hogben, The minimum rank of symmetric matrices described by a graph: A survey, *Linear Algebra Appl.* 426 (2007) 558–582.

Acknowledgement

- Mentors: Dr. Leslie Hogben, Dr. M. Reza Emamy-K.
- Research Assistant: Jason Smith
- National Science Foundation-
  - Alliance Grant DMS:0502354