Minimum Rank of Subgraphs of Hypercubes

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Matrices and Graphs

Definition

The graph $G(A) = (V, E)$ of $n \times n$ matrix $A$ is a graph where:

1. $V = \{1, \ldots, n\}$
2. $E = \{ij : a_{ij} \neq 0\}$
3. Diagonal of $A$ is ignored

Example:

\[
\begin{bmatrix}
3 & 1 & 0 & 0 \\
1 & 0 & 2 & 3 \\
0 & 2 & -1 & -5 \\
0 & 3 & -5 & 2
\end{bmatrix}
\]
Definition

The set of symmetric matrices described by a graph $G$ (over $\mathbb{R}$) is $S(G) = \{ A \in S_n(\mathbb{R}) : G(A) = G \}$

Example:

\[
\begin{bmatrix}
? & a & b & c \\
a & ? & 0 & 0 \\
b & 0 & ? & 0 \\
c & 0 & 0 & ?
\end{bmatrix}
\]
**Minimum Rank**

**Definition**

The *minimum rank of* $G$ is $mr(G) = \min\{\text{rank}(A) \mid A \in \mathcal{S}(G)\}$.

**Example:**

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
$$

$$mr(K_4) = 1$$
Maximum Nullity

**Definition**

The maximum nullity of $G$ is $M(G) = \max\{\text{corank}(A) : A \in S(G)\}$

**Example:**

$$M(K_4) = 3$$
**Maximum Nullity**

**Definition**

The maximum nullity of $G$ is $M(G) = \max\{\text{corank}(A) : A \in S(G)\}$

**Example:**

1. $M(K_4) = 3$
2. $mr(K_4) = 1$
3. $|K_4| = 4$

Note that $M(K_4) = |K_4| - mr(K_4)$
**Induced Subgraph**

**Definition**

A graph $G' = (V', E')$ is a *subgraph* of graph $G = (V, E)$ if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$.

**Example:**

$$G = \begin{array}{c}
\begin{array}{c}
\text{Graph 1} \\
\text{Graph 2} \\
\text{Graph 3}
\end{array}
\end{array}$$
Induced Subgraph

**Definition**

A graph $G' = (V', E')$ is a **subgraph** of graph $G = (V, E)$ if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$.

**Example:**

$G = \quad$ (Graph)

**Definition**

A subgraph $G[V']$ is an **induced subgraph** of $G$ by $V'$ if $\forall v_i, v_j \in V'$, $v_iv_j \in E(G) \Rightarrow v_iv_j \in E(G[V'])$. 
Observation

1. If $G'$ is an induced subgraph of $G$, then $mr(G') \leq mr(G)$.
2. $mr(G) + M(G) = |G|$
3. If $G = \bigcup_{i=1}^{h} G_i$ then $mr(G) \leq \sum_{i=1}^{h} mr(G_i)$. 
Coloring

Color Change Rule

If \( G \) is a graph with each vertex colored either white or black, \( u \) is a black vertex of \( G \), and exactly one neighbor \( v \) of \( u \) is white, then change the color of \( v \) to black.

Example:
Color Change Rule

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Example:
Zero Forcing Set and Number

**Definition**

- **Derived coloring** is the result of applying the color-change rule until no more changes are possible.
- **Zero forcing set (ZFS)** $Z \subseteq V$, s.t. if initially $v_i \in Z$ are colored black and $v_j \notin Z$ are colored white, then the derived coloring is all black.
- **Zero forcing number** $Z(G) = \min\{|Z| : Z \text{ is a ZFS}\}$.

**Theorem (AIM08)**

For any graph $G$, $M(G) \leq Z(G)$.
Hypercube

**Definition**

The *Cartesian Product* of two graphs $G$ and $H$, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such that $(u, v)$ is adjacent to $(u', v')$ if and only if:

1. $u = u'$ and $vv' \in E(H)$, or
2. $v = v'$ and $uu' \in E(G)$

**Example:** $P_3 \square P_2

![Diagram of $P_3 \square P_2$]
**Definition (Hypercube)**

\[ Q_0 = (V, E) \text{ where } |V| = 1, |E| = 0 \]

\[ Q_d = Q_{d-1} \square P_2 \]

**Example:** \[ Q_4 = Q_3 \square P_2 \]
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Hypercube

Definition (Cut Complex)

A Cut Complex $\mathcal{C}$ is a subgraph of $Q_d$ for which there is a $(d - 1)$-dimensional hyperplane $\mathcal{H}$ that strictly separates the vertices of $\mathcal{C}$ from the rest of the vertices of $Q_d$.

Example:
**Definition (Cut Complex)**

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**Example:**

![Diagram of a Cut Complex]

[Hypercube]

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**Definition (Cut Complex)**

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**Example:**

![Diagram of a cut complex](image)
**Definition (Cut Complex)**

A **Cut Complex** $C$ is a subgraph of $Q_d$ for which there is a $(d - 1)$-dimensional hyperplane $\mathcal{H}$ that strictly separates the vertices of $C$ from the rest of the vertices of $Q_d$.

**Example:**

1. Red vertices are in $C_0$.
2. Blue vertices are in $C'_0$.
3. Cut complexes are induced subgraph of $Q_d$. 
Definition

The path cover number of $G, P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of $G$ that cover all the vertices of $G$; such a set of paths realizing $P(G)$ is called a minimal path cover.

Theorem (Johnson, Leal Duarte 99)

Let $T$ be a tree,
$M(T) = P(T) = |T| - mr(T)$
Theorem

Let $H$ be an induced subgraph of $Q_d = Q_{d-1} \square P_2$ such that $Q_{d-1} \subseteq H$ and $H$ contains $Q_{d-2}$ from the other copy of $Q_{d-1}$. Then $\text{mr} (H) = \text{mr} (Q_d) = 2^{d-1}$ and $M(H) = Z(H) = |H| - 2^{d-1}$.

Example:
Example:

- $H$ is an induced subgraph of $Q_4$
  - $\text{mr}(H) \leq \text{mr}(Q_4)$
  - $\text{mr}(H) \leq 8$
- Order of graph: $|H| = 15$
- Zero forcing number: $Z(H) \leq 7$
- $M(H) \leq Z(H) \Rightarrow M(H) \leq 7$
- $\text{mr}(H) + M(H) = |H|$
  - $\Rightarrow \text{mr}(H) = |H| - M(H)$
  - $\Rightarrow \text{mr}(H) \geq 15 - 7 = 8$
  - $\Rightarrow \text{mr}(H) = 8 = \text{mr}(Q_4)$
- $M(H) = |H| - \text{mr}(H)$
- $M(H) = 7 = Z(H)$
Conjecture

If $H$ is a cut-complex of $Q_d$ then $M(H) = Z(H)$.

For $d = 1, 2, 3, 4$ and $H$ a cut-complex of $Q_d$; $mr(H), M(H), Z(H)$ have been computed.
Remark

There is an example of an induced subgraph $H$ of $Q_5$ with $M(H) < Z(H)$. $H$ is not a cut-complex.


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