Minimum Rank of $n$-Dimensional Hypercube Cut-Complex

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**Basic Definitions**

**Definition**

A **graph** $G$ is an ordered tripled $(V_G, E_G, \psi_G)$ consisting of a non-empty set of vertices denoted by $V_G$ and a set $E_G$, disjoint from $V_G$, of edges and an incidence function $\psi_G$ that associate with each edge of $G$ an unordered pair of vertices of $G$, note that this unordered pair of vertices are not necessarily distinct.

**Definition**

A graph is **finite** if both its vertex set and edge set are finite.

**Definition**

The **order** of a graph $G$ is defined by the number of vertices in $V_G$ and denoted by $|G|$.
Isomorphic Graphs

Definition

Two graphs $G$ and $H$ are isomorphic ($G \cong H$) if there exist a one-to-one correspondence between their vertex set which preserves adjacency.
**Induced Subgraphs**

**Definition**

A graph $G' = (V', E')$ is a **subgraph** of graph $G = (V, E)$ if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$.

![Graphs](image)

**Figure:** Graph, Induced subgraph, subgraph

**Definition**

A subgraph $H$ of a graph $G$ is said to be **induced** if for any pair of vertices $v_i$ and $v_j$, $\{v_i, v_j\}$ is an edge of $H$ if and only if $\{v_i, v_j\}$ is an edge of $G$. 
Graphs Operations

Definition

- The complement $\overline{G}$ of a graph $G$ also has $V(G)$ as its vertex set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$, denoted by $\overline{G} = (V_G, \overline{E}_G)$.

- The union of graphs $G_i = (V_{G_i}, E_{G_i})$ is defined by

$$\bigcup_{i=1}^{n} G_i = \left( \bigcup_{i=1}^{n} V_{G_i}, \bigcup_{i=1}^{n} E_{G_i} \right) .$$
Graphs Operations

Definition

The **Cartesian product** of two graphs \( G \) and \( H \) \( G \Box H \) is a graph such that:

- the vertex set of \( G \Box H \) is the Cartesian product \( V(G) \times V(H) \); and
- any two vertices \((u, u')\) and \((v, v')\) are adjacent in \( G \Box H \) if and only if either
  - \( u = v \) and \( u' \) is adjacent with \( v' \) in \( H \), or
  - \( u' = v' \) and \( u \) is adjacent with \( v \) in \( G \).
Example of Cartesian product

Figure: $P_3 \Box P_2$
We can define a hypercube using a Cartesian product,

**Definition**

Let $Q_0 = (V, E)$ where $|V| = 1, |E| = 0$. We can define a $n$-dimensional hypercube $Q_n = Q_{n-1} \square P_2$.

The **hypercube graph** $Q_n$ is a regular graph (that is, each vertex of $Q_n$ is incidence to exactly $n$ edges) with $2^n$ vertices which correspond to the subsets of a set with $n$ elements.

**Example** ($Q_4 = Q_3 \square P_2$)
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A Cut Complex $C$ is a subgraph of $Q_n$ for which there is a $(n - 1)$-dimensional hyperplane $H$ that strictly separates the vertices of $C$ from the rest of the vertices of $Q_n$. 
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**Definition**

A **Cut Complex** $\mathcal{C}$ is a subgraph of $Q_n$ for which there is a $(n - 1)$-dimensional hyperplane $\mathcal{H}$ that strictly separates the vertices of $\mathcal{C}$ from the rest of the vertices of $Q_n$.

1. **Red** vertices are in $\mathcal{C}_0$.
2. **Blue** vertices are in $\mathcal{C}_0'$.
3. Cut complexes are induced subgraph of $Q_n$. 
Remark

Note that not all induced subgraph are a cut-complex but all cut-complex is an induced subgraph.

Figure: Robot Graph
### Definition

The graph $G(A) = (V, E)$ of $n \times n$ matrix $A$ is a graph where:

1. $V = \{1, \ldots, n\}$
2. $E = \{ij : a_{ij} \neq 0\}$
3. Diagonal of $A$ is ignored

### Example

$$
\begin{bmatrix}
3 & 1 & 0 & 0 \\
1 & 0 & 2 & 3 \\
0 & 2 & 1 & 0 \\
0 & 3 & 2 & 0 \\
\end{bmatrix}
$$

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**Symmetric Matrices**

**Definition**

The set of symmetric matrices described by a graph $G$ (over $\mathbb{R}$) is $S(G) = \{ A \in S_n(\mathbb{R}) : G(A) = G \}$

**Example**

\[
\begin{bmatrix}
? & a & b & c \\
 a & ? & 0 & 0 \\
b & 0 & ? & 0 \\
c & 0 & 0 & ?
\end{bmatrix}
\]

Note that $? \in \mathbb{R}$ and $a, b, c \in \mathbb{R} \setminus \{0\}$. 
Theorem (Theorem 2.2 in Zhang)

Let $M$ be a square matrix partitioned as:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Then $\det(M) = \det(AD - CB)$, if $AC = CA$.

Definition

Define the **Kronecker product** $A \otimes B$ of two matrices $A$ and $B$ to be the matrix we get by replacing the $ij$ - entry of $A$ by $a_{ij}B$, for all $i$ and $j$.

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$
For the Cartesian product of two graphs $G$ and $H$ we can define the adjacency matrix as (see Section 9 in Godsil & Royle):

$$A(G \square H) = A(G) \otimes I + I \otimes A(H)$$

where $I$ is the identity matrix.

**Theorem**

Let $G$ and $H$ be a graphs with $n$ and $k$ vertices respectively. Let $A \in S(G)$ and $B \in S(H)$. Then $A \otimes I_n + I_k \otimes B \in S(G \square H)$. 

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**Minimum Rank**

**Definition**

The *minimum rank of G is* \( mr(G) = \min \{ \text{rank}(A) \mid A \in S(G) \} \).

**Example**

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

\[mr(K_4) = 1\]

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Maximum Nullity

**Definition**

The *maximum nullity* of $G$ is defined by:

$$M(G) = \max \{ \text{corank}(A) : A \in S(G) \}$$

**Example**

1. $M(K_4) = 3$

Note that $M(K_4) = |K_4| - \text{mr}(K_4)$
Definition

The maximum nullity of $G$ is defined by:

$$M(G) = \max \{ \text{corank}(A) : A \in S(G) \}$$

Example

1. $M(K_4) = 3$
2. $mr(K_4) = 1$
3. $|K_4| = 4$

Note that $M(K_4) = |K_4| - mr(K_4)$
Minimum Rank and Maximum Nullity Properties

The following properties are well known and straightforward:

**Proposition**

1. If $G'$ is an induced subgraph of $G$, then $mr(G') \leq mr(G)$.
2. $mr(G) + M(G) = |G|$
3. If $G = \bigcup_{i=1}^{h} G_i$ then $mr(G) \leq \sum_{i=1}^{h} mr(G_i)$.
4. Let $T$ be a tree. $M(T) = |T| - mr(T)$.
5. Let $C_n$ be a $n$-cycle. $mr(C_n) = n - 2$.
6. Let $P_n$ be a $n$-path. $mr(P_n) = n - 1$. 
Theorem (2.8 Survey07)

If $G$ is a connected graph, $mr(G) \leq 2$ if and only if $G$ does not contain as an induced subgraph any of $P_4$, Dart, $\times$, or $K_{3,3,3}$.

Figure: $P_4$, Dart, $\times$, $K_{3,3,3}$
Color Change Rule

If $G$ is a graph with each vertex colored either white or black, $u$ is a black vertex of $G$, and exactly one neighbor $v$ of $u$ is white, then change the color of $v$ to black.

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Example
Zero Forcing Set and Number

**Definition**

- **Derived coloring** is the result of applying the color-change rule until no more changes are possible.
- **Zero forcing set (ZFS)** $Z \subseteq V$, s.t. if initially $v_i \in Z$ are colored black and $v_j \notin Z$ are colored white, then the derived coloring is all black.
- **Zero forcing number** $Z(G) = \min\{|Z| : Z \text{ is a ZFS}\}$.

**Theorem (Theorem 2.4 AIM08)**

*For any graph $G$, $M(G) \leq Z(G)$.***
Theorems of Zero Forcing Number

**Theorem (Proposition 2.5 AIM 08)**

*For any graphs G and H,*

\[ Z(G \sqcup H) \leq \min\{Z(G) | H|, Z(H) | G|\}. \]

The following corollaries are in consequence of Theorem

**Corollary (Corollary 2.6 AIM08)**

\[ Z(G \sqcup P_n) \leq \min\{|G|, Z(G) n\}. \]

**Corollary (Corollary 2.7 AIM08)**

\[ Z(Q_n) \leq 2^{n−1} \]

**Theorem (Theorem 3.1 AIM08)**

*For the n-dimensional hypercube, M(Q_n) = Z(Q_n) = 2^{n−1}.*
mr \( (G) \) using induced subgraph

Let \( G = C_10' \), note that \( |G| = 10 \) and \( Z(G) \leq 4 \) then by Theorem 2.4AIM08 \( M(G) \leq 4 \) then \( mr(G) \geq 6 \). Moreover, \( G \) is an induced subgraph of \( C_{11}' \) and by Theorem of subgraphs (Survey07) then:

\[
\begin{align*}
mr(G) & \leq mr(C_{11}') \\
mr(G) & \leq 6
\end{align*}
\]

since \( mr(G) \leq 6 \) and \( mr(G) \geq 6 \) then \( mr(G) = 6 \).
Let $T = C'_5$, note that $|T| = 5$ and $Z(T) \leq 3$, by Theorem 2.4 (AIM08) we obtain that $M(T) \leq 3$. By Theorem of tree (Survey07) we have $\text{mr}(T) \geq |T| - M(T) \geq 2$. Since $T$ is a connected graph by Theorem 2.1.6 we have that $\text{mr}(T) \leq 2$ then $\text{mr}(T) = 2$. 

mr $(G)$ using trees
Let \( G = C_{10} \), note that \( |G| = 10 \) and \( Z(G) \leq 4 \) then by AIM08 \( M(G) \leq 4 \) so \( mr(G) \geq 6 \). Let \( G_1 \) the graph with blue edges (\( C_4 \)) so, \( mr(G_1) = 2 \) and let \( G_2 \) the graph with red edges as \( G_2 \) (\( mr(G_2) = 4 \)), note that \( G = G_1 \cup G_2 \). By Proposition (Survey07)

\[
\begin{align*}
    mr(G) & \leq mr(G_1) + mr(G_2) \\
    mr(G) & \leq 2 + 4 \\
    mr(G) & \leq 6
\end{align*}
\]

since \( mr(G) \leq 6 \) and \( mr(G) \geq 6 \) then \( mr(G) = 6 \).
mr (G) using Mathematica and SAGE

Figure: $C'_{11}$ with an induced path
**Theorem**

Let $H$ be an induced subgraph of $Q_n = Q_{n-1} \boxtimes P_2$ such that $Q_{n-1} \subseteq H$ and $H$ contains $Q_{n-2}$ from the other copy of $Q_{n-1}$. Then $mr(H) = mr(Q_n) = 2^{n-1}$ and $M(H) = Z(H) = |H| - 2^{n-1}$.

**Proof.**

Let $\bar{f} = Q_{n-1}$ such that $\bar{f} \subseteq H$ and $H$ contains $Q_{n-2} \subseteq f$. Since $H$ be induced by $Q_n$ then $mr(H) \leq mr(Q_n) = 2^{n-1}$. Exhibiting ZFS of $|H| - |\bar{f}|$, then we obtain $Z(H) = |H| - 2^{n-1}$ We know that

$$
mr(H) + M(H) = |H|
$$

$$
mr(H) = |H| - M(H) \geq |H| - (|H| - 2^{n-1})
$$

$$
mr(H) \geq 2^{n-1}
$$

therefore $mr(H) = 2^{n-1}$.
Faces Theorem

- $H$ is an induced subgraph of $Q_4$
  - $\text{mr}(H) \leq \text{mr}(Q_4)$
  - $\text{mr}(H) \leq 8$
- Order of graph: $|H| = 15$
- Zero forcing number: $Z(H) \leq 7$
- $M(H) \leq Z(H) \Rightarrow M(H) \leq 7$

- $\text{mr}(H) + M(H) = |H|$
  - $\Rightarrow \text{mr}(H) = |H| - M(H)$
  - $\Rightarrow \text{mr}(H) \geq 15 - 7 = 8$
  - $\Rightarrow \text{mr}(H) = 8 = \text{mr}(Q_4)$
- $M(H) = |H| - \text{mr}(H)$
- $M(H) = 7 = Z(H)$
Theorem

\[ M(Q_n \Box P_3) = Z(Q_n \Box P_3) = 2^n \]

Proof: We know that \( \text{mr}(Q_n) = 2^{n-1} \). Note that \( Q_n \Box P_3 \) can be embedded in \( Q_{n+2} \) then:

\[
\text{mr}(Q_n \Box P_3) \leq \text{mr}(Q_{n+2}) = 2^{(n+2)-1} \leq 2^{n+1} = 2 \cdot 2^n
\]

By Theorem 2.6 (AIM08) we have:

\[
Z(Q_n \Box P_3) \leq \min \{|Q_n|, Z(Q_n)\} \leq \min \{2^n, 2^{n-1} \cdot 3\} \leq 2^n
\]

Then \( M(Q_n \Box P_3) \leq 2^n \).
Since $|Q_n \square P_3| = 3 \cdot 2^n$,

\[
\begin{align*}
M(Q_n \square P_3) &= |Q_n \square P_3| - \text{mr}(Q_n \square P_3) \\
M(Q_n \square P_3) &\geq 3 \cdot 2^n - 2 \cdot 2^n \\
M(Q_n \square P_3) &\geq 2^n (3 - 2) \\
M(Q_n \square P_3) &\geq 2^n
\end{align*}
\]

Then $M(Q_n \square P_3) = 2^n$.
### Table with results

| Cut-complex | $|G|$ | $mr(G)$ | $M(G)$ | $Z(G)$ |
|-------------|------|---------|--------|--------|
| $C_1$       | 1    | 0       | 1      | 1      |
| $C_2$       | 2    | 1       | 1      | 1      |
| $C_3$       | 3    | 2       | 1      | 1      |
| $C_4$       | 4    | 2       | 2      | 2      |
| $C_5$       | 5    | 3       | 2      | 2      |
| $C_6$       | 6    | 4       | 2      | 2      |
| $C_7$       | 7    | 4       | 3      | 3      |
| $C_8$       | 8    | 4       | 4      | 4      |
| $C_8'$      | 8    | 5       | 3      | 3      |
| $C_7'$      | 7    | 5       | 2      | 2      |
| $C_7''$     | 8    | 5       | 3      | 3      |
| $C_8''$     | 8    | 5       | 3      | 3      |
| $C_6'$      | 6    | 4       | 2      | 2      |
| $C_4'$      | 4    | 2       | 2      | 2      |
| $C_5'$      | 5    | 2       | 3      | 3      |
Table with results

<table>
<thead>
<tr>
<th>Cut-complex</th>
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<th>mr (G)</th>
<th>M (G)</th>
<th>Z (G)</th>
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</tbody>
</table>
Conjecture

If $H$ is a cut-complex of $Q_n$ then $M(H) = Z(H)$.

For $d = 1, 2, 3, 4$ and $H$ a cut-complex of $Q_n$; $mr(H), M(H), Z(H)$ have been computed.


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